Quantum Logic

In the quantum theory classical patterns of inference break down. Mathematical structures called lattices can model alternative roles for the words "and" and "or" that may fit the world more coherently

by R. I. G. Hughes

If Aunt Agatha is dead and either the butler did it or the gardener did it, then surely either Aunt Agatha is dead and the butler did it or Aunt Agatha is dead and the gardener did it. If a quarter is in a box and it shows either heads or tails, then either the quarter is in the box and it shows heads or the quarter is in the box and it shows tails. The reasoning in such sentences compels assent, although the conclusions seem rather trivial, given the premises. Moreover, the subject matter appears to be entirely incidental to the validity of the inferences. Neither the wealth of Aunt Agatha nor the greed of the butler nor the concept of death has any bearing on the conclusion reached in the first argument; neither the monetary value of a quarter nor the definition of heads or tails could alter the conclusion of the second argument. Because of the irrelevance of the subject matter the validity of the inferences depends only on the rules of logic. In particular it depends only on the structural patterns displayed by connectives such as *and* and *or*.

Consider now certain physical phenomena of microscopic scale as they are described by the quantum theory. According to the theory, the electron (like a number of other elementary particles) has an intrinsic angular momentum, or spin. The spin is quantized: it is always found to assume one of only two values, either up or down, along any direction in which it is measured. It is impossible, however, to specify the spin of an electron along two spatial axes simultaneously. For example, if the spin of an electron measured along the x axis is up, it is not possible to assign any definite value to the spin along the y axis.

Suppose a beam of electrons is completely spin-polarized along the x axis, which means that all the electrons in the beam are found to have the same spin value (say spin-up) whenever the spin is measured along the x axis. Because the beam has not been polarized along the y axis, one can say of each electron in the prepared beam that its spin along the x axis is up and that its spin along the y axis is either up or down. Following the pattern of reasoning I employed in analyzing Aunt Agatha's murder, the statement about the electron implies either that the spin along the x axis is up and the spin along the y axis is up or that the spin along the x axis is up and the spin along the y axis is down.

Both clauses of this assertion, however, violate the principle of quantum mechanics stating that spin cannot be specified simultaneously along two axes. Since neither clause can be accepted, the assertion itself must be rejected. One must therefore either reject the initial statement about the prepared beam of electrons or disallow a logical procedure for defining the consequences of the statement, a procedure that seemed quite innocuous in ordinary reasoning. There is no motive at hand for rejecting the initial statement, and so it seems at least one law of classical logic cannot be applied to quantum phenomena.

Any proposal to revise the laws of logic, or even to regard them as being open to revision, runs counter to centrally and universally held beliefs. Nevertheless, the alternative of retaining classical logic by denying the validity of the quantum theory has little appeal. Fifty years after its inception quantum mechanics is one of the most successful of scientific theories. Its versatility and its predictive power are such that it has no serious rivals. It is applied routinely and with high precision to the understanding of the interactions of elementary particles. The theory also provides a means of describing a wide range of other phenomena, including the physics and the chemistry of atoms, molecules and solid materials.

In the opinion of some philosophers of science there is precedent in the history of physics for the kind of conceptual shift entailed by a modification of classical logic. It is now commonplace to point out that in the development of physics since Einstein the notions of space, time, energy, momentum and mass have been profoundly altered. The change can be expressed in what philosophers call the formal mode of speech by stating that the roles of words such as *space, time, energy, momentum* and *mass* are different in the language of modern physics from what they were in the language of classical physics. Similarly, one might suggest that if the conceptual structure of physics is embodied in the language of physics, then even the roles of words such as *and, or* and *not* are not exempt from revision. Because these words have traditionally been associated with logical investigations, and because the shift in their roles is motivated by the development of quantum mechanics, it makes sense to call the result of the shift quantum logic.

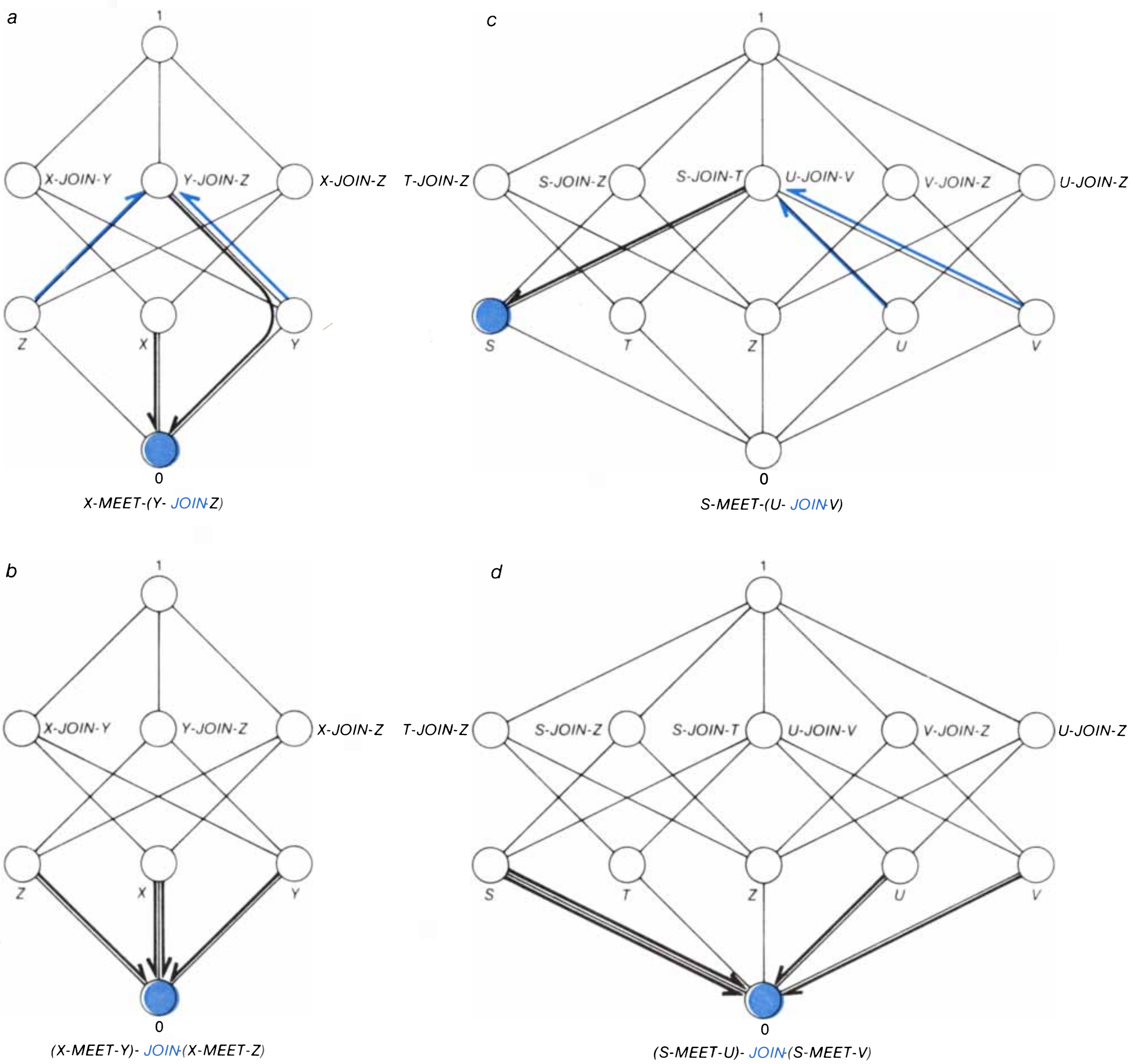
How might one characterize the role played in a language by the words that serve as logical connectives? Suppose P designates the sentence "Aunt Agatha is dead," Q designates the sentence "The butler did it" and R designates the sentence "The gårdener did it." Then the logical structure of my first inference can be set forth as follows:

From *P and* (*Q or R*)
one infers (*P and Q*) *or* (*P and R*).

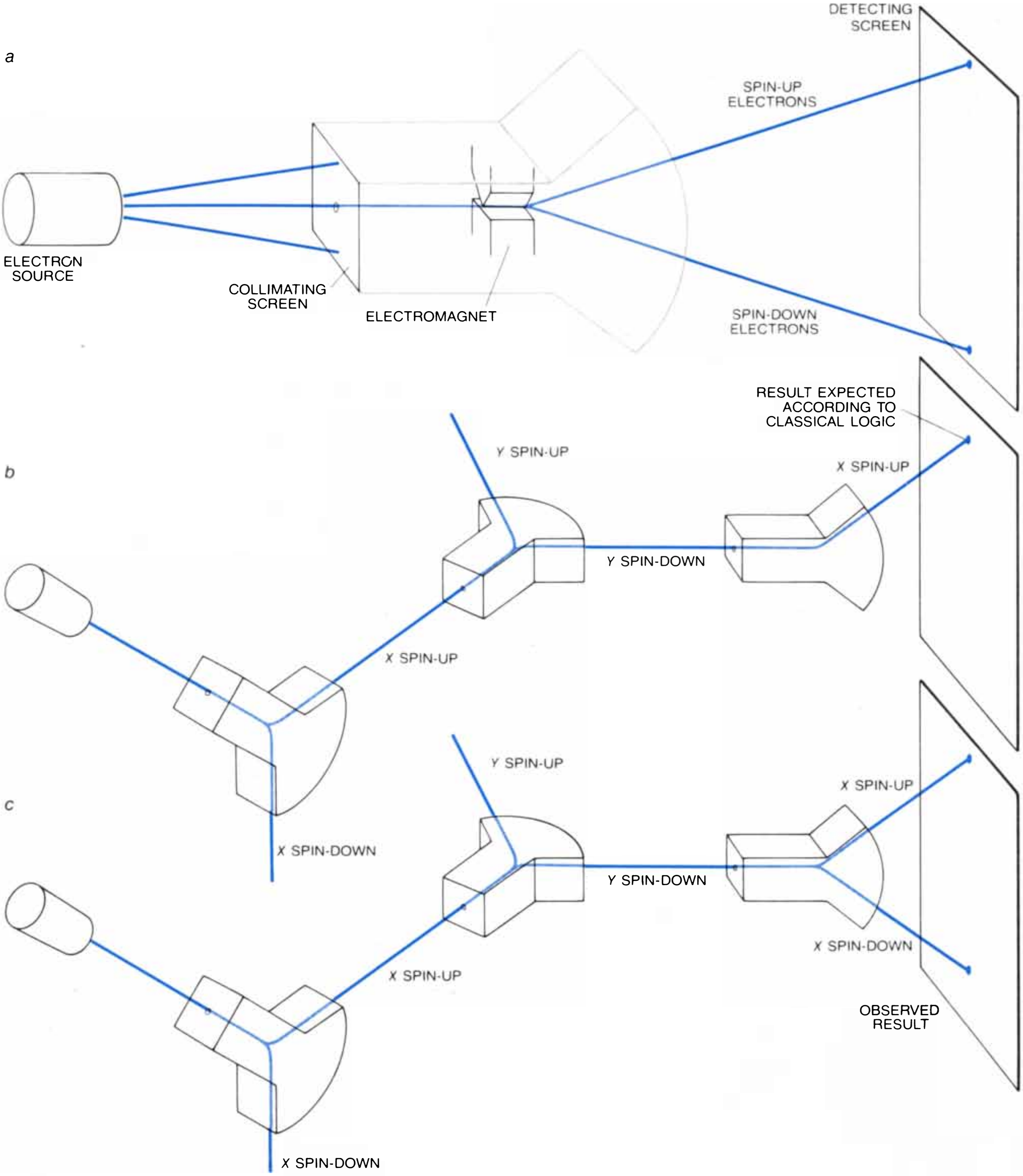
Writing the logical form of the reasoning in this way is meant to indicate that the inference is valid no matter what simple declarative sentences are substituted for the letters P, Q and R. The formula is called a distributive law of logic. In another distributive law of logic the words *and* and *or* are interchanged. Such laws of logic closely resemble the distributive law of arithmetic, which states, for instance, that the expression $2 \times (3 + 4)$ is equal to $(2 \times 3) + (2 \times 4)$.

In classical logic the roles of the logical connectives *and* and *or* are at least partially and implicitly defined by the patterns they form in the distributive law. In the quantum-mechanical description of electron spin, however, the logical step from the first line of the distributive law to the second line is disallowed. Suppose *P* designates the sentence "The spin along the *x* axis is up," *Q* designates the sentence "The spin along the *y* axis is up" and *R* designates the sentence "The spin along the *y* axis is down." The formula *P and* (*Q or R*) may then be true, but the formula (*P and Q*) *or* (*P and R*) cannot be maintained. Hence the suggestion that quantum mechanics may demand the revision of a law of logic amounts to the proposal that the roles of the connectives *and* and *or* must be altered so that statements about quantum mechanics no longer combine logically to satisfy the distributive law.

How can one be sure that such a change in the language preserves some sense of the traditional logical structure and does not lead to inconsistency or paradox? The traditional response to the question has been to build a mathematical model that incorporates the change but still exhibits reasonable behavior. (What is meant by "reasonable behav-



DISTRIBUTIVE LAW OF LOGIC states that for any sentences *P*, *Q* and *R*, if the compound sentence *P and* (*Q or R*) is true, then the compound sentence (*P and Q*) *or* (*P and R*) must also be true. The distributive law can be modeled by a lattice: an array of points and a network of lines that connect lower points with upper ones. The points represent sentences and the lines represent entailment relations. A sentence represented by a given point entails all the sentences represented by points higher in the lattice that can be reached from the given point along upward-moving lines. Two operations called *meet* and *join* can be defined on the lattice. The *meet* of two points is the highest point to which they are both connected by lines moving downward from at least one of them. The *join* of two points is the lowest point to which they are both connected by lines moving upward from at least one of them. If the operation *meet* is identified with the word *and* and the operation *join* is identified with the word *or*, the lattice can model the logical relations among sentences. The structure of the lattice determines whether or not distributive relations hold for the operations *meet* (*black arrows*) and *join* (*colored arrows*). For the lattice in panels *a* and *b* the operations *x-meet-*(*y-join-z*) and (*x-meet-y*)*-join-*(*x-meet-z*) lead to the same point, so that the operations satisfy a form of the distributive law. Such a lattice models classical, or distributive, logic. For the lattice in panels *c* and *d* the two operations do not lead to the same point. The lattice models the nondistributive logic that seems to be required for the description of quantum-mechanical phenomena. (The *join* operation in panels *b* and *d* is not shown by an arrow because it does not lead away from the bottom point, 0.)



OBSERVED PROPERTIES OF SPIN, or intrinsic angular momentum, suggest that the distributive law of logic cannot be applied to the description of an atomic or a subatomic particle. According to classical mechanics, particles whose spins are randomly oriented should be deflected along a continuous range of paths by a magnetic field that varies from point to point. Otto Stern and Walther Gerlach found in 1921 that such a field deflects particles along only two paths, indicating that the spin is quantized (*a*); the component of the spin measured along any designated axis is invariably either up or down. In each of the deflected beams all the particles have a spin-up component along the axis of the magnetic field (say the *x* axis), and they have either a spin-up or a spin-down component along an axis perpendicular to the field (say the *y* axis). In classical logic this is equivalent to saying that some particles in the deflected beam are *x* spin-up and *y* spin-up and some are *x* spin-up and *y* spin-down. A second magnetic field at right angles to the first field can split one of the beams into its two *y* components. One would expect that a third magnetic field, oriented along the *x* axis like the first field, would give rise to only one beam, since the particles had previously been exposed to spin-polarizing fields first along the *x* axis and then along the *y* axis (*b*). Instead two beams emerge from the third field; each beam is polarized along the *x* axis, but the *y* component is randomly oriented (*c*). The result indicates it is impossible to simultaneously assign exact values to the *x* and *y* components of the spin of a particle. The equivalence derived from the distributive law of classical logic does not hold because the *x* and the *y* spins cannot be defined simultaneously.

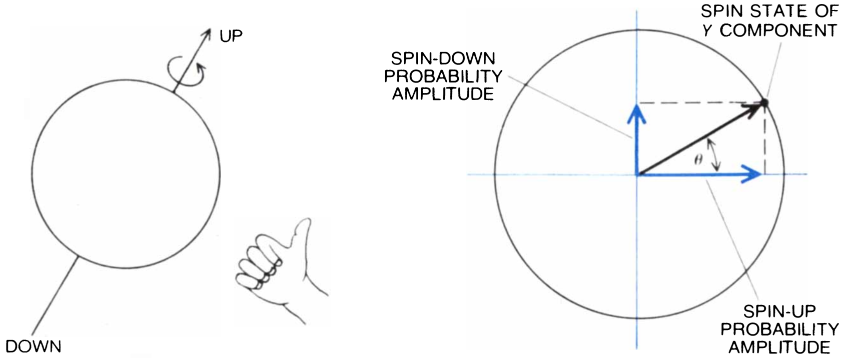
ior" probably cannot be specified in advance.) An instructive example of this process is provided by the development of non-Euclidean geometry.

The development began when mathematicians came to doubt the self-evidence of the fifth postulate of Euclid. The fifth postulate states that through any point in a plane that does not lie on a given line one and only one line can be drawn that is parallel to the given line. By denying the fifth postulate and assuming instead that either many parallel lines or no parallel lines can be drawn through the given point, 19th-century geometers were able to construct rich formal systems. The systems were composed of postulates and theorems that at first could not be interpreted as statements about geometry at all.

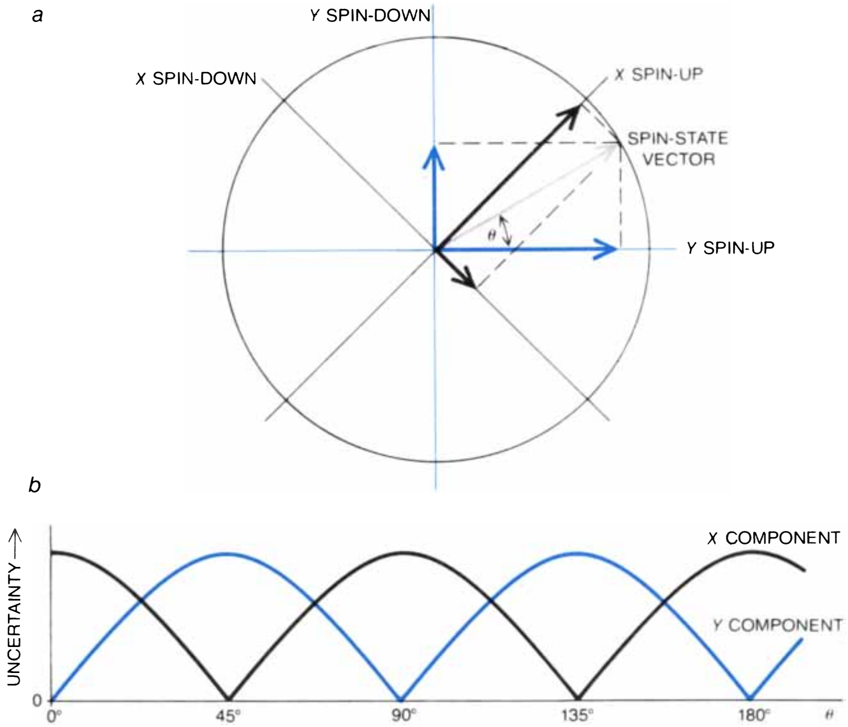
In spite of the intrinsic interest of such formal systems to mathematicians, it is difficult to work with the systems if there is no way to interpret their theorems and postulates. Initially geometers could not guarantee that all their results were free of inconsistencies, nor did they perceive that the various formal systems are deeply related. After some time, however, it was recognized that statements in the formal systems could be interpreted as statements about the geometry of curved surfaces, such as the surface of a sphere or the saddle-shaped infinite hyperbolic surface.

The construction of geometric models made possible a more intuitive understanding. Abstract patterns in the formal systems could be visualized as geometric relations, and one could see at a glance whether or not a theorem made sense. Moreover, mathematicians came to regard each geometric model as being merely one case in the more general study of curved surfaces; the geometry of a particular model could be characterized by specifying the curvature of the surface. Euclidean geometry, for example, is the study of flat surfaces, that is, surfaces whose curvature is zero. In this way the geometric interpretation of the various formal systems showed that they are not unrelated but are members of a single family. The theorems within each system came to be understood as analogues of one another, playing similar roles in the geometries of different surfaces.

In 1932, in order to construct a similar interpretive model for quantum logic, John von Neumann of the Institute for Advanced Study began investigating the properties of mathematical structures called lattices. Von Neumann elaborated his approach in 1936 in an article written with Garrett Birkhoff of Harvard University. They showed that the lattice structure of a physical theory can be regarded as a mathematical model of the system of logic appropriate to the theory. The notion of a lattice is quite



SPIN STATE OF A PARTICLE gives the probability that the spin measured along a given spatial axis is either up or down. By convention, if the fingers of the right hand curl in the same sense as the spin, the right thumb points in the spin-up direction. The probability of finding that the particle is spin-up along a given axis is the square of the magnitude of a vector called the spin-up probability amplitude. Similarly, the probability of finding that the particle is spin-down along a given axis is the square of the spin-down probability amplitude. The probability that the particle is either spin-up or spin-down is 1, so that the sum of the squares of the two probability amplitudes is also 1. If the probability amplitudes are graphed at right angles to each other, their vector sum is the hypotenuse of a right triangle; the length of the hypotenuse is 1. Hence any spin state of a particle can be represented by a vector that is the vector sum of the two probability amplitudes, and the set of all possible spin states corresponds to a circle of radius 1 in an abstract space called phase space. A complete analysis of spin states requires the introduction of complex numbers, that is, numbers with both a real and an imaginary part.

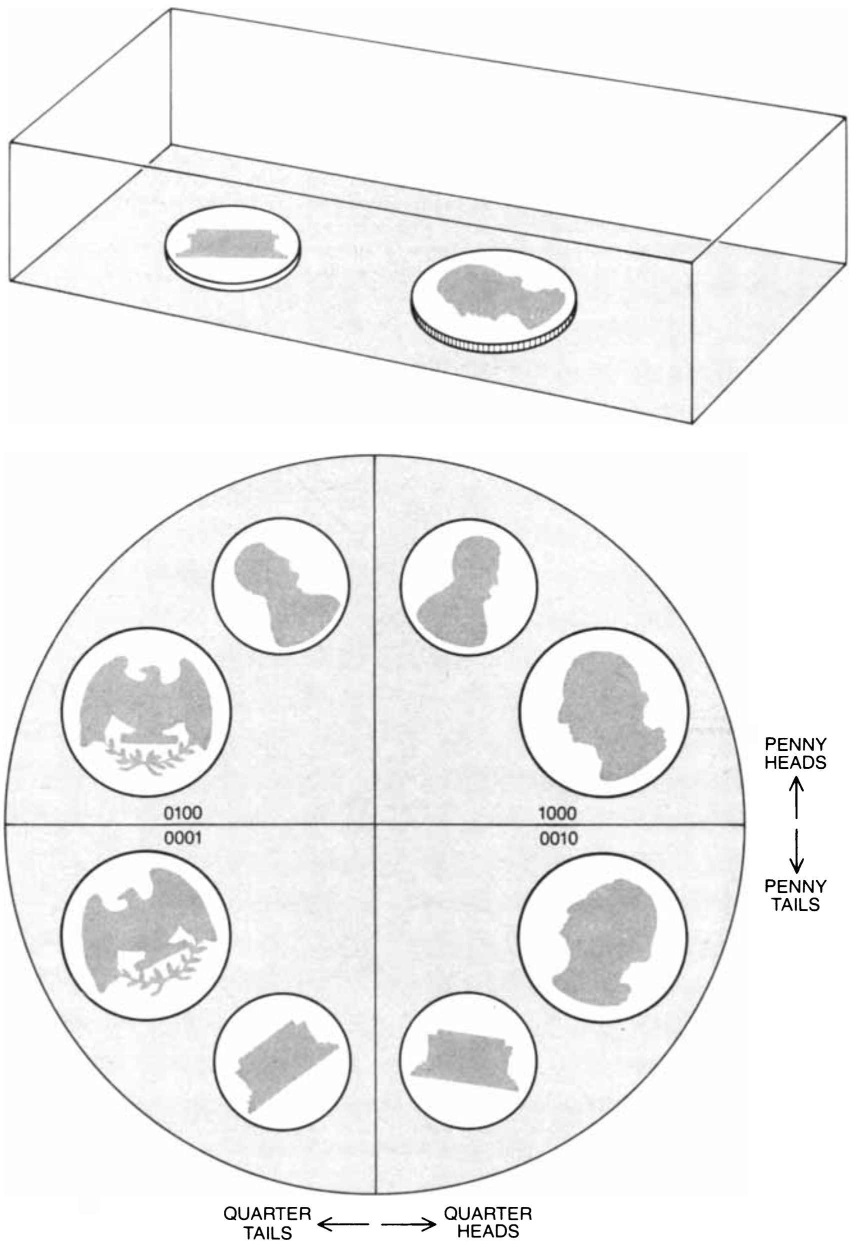


UNCERTAINTY PRINCIPLE of Werner Heisenberg states that the values of specified pairs of variables that characterize the state of a particle cannot be known simultaneously with unlimited precision. The *x* and *y* components of the spin of a particle form such a pair: they are said to be incompatible. Here the spin-up and spin-down probability amplitudes are given for the *x* and *y* components of the spin (*a*). The spin-state vector (*gray arrow*) can be treated as a vector sum of either the two *x*-component amplitudes (*black arrows*) or the two *y*-component amplitudes (*colored arrows*). The diagram is only an approximation; a full representation of the spin state would have to include the *z* component and could be drawn only in a space whose points represent complex numbers. As the vector rotates away from the *y* spin-up axis to the *x* spin-up axis the probability of finding that the *x* component of spin is up increases, but the result of any measurement of the *y* component becomes more uncertain. By imagining that the spin-state vector continues to be rotated counterclockwise, a graph of the uncertainties associated with the two spin components can be constructed (*b*). When the uncertainty in the value of one component falls to zero, the uncertainty in the value of the other component is maximal.

general. Here I shall apply it to the logical structure of a simple theory contrived for the purpose of illustration, but it can also capture the structures of classical physics and quantum mechanics. In the past dozen years physicists and philosophers have been returning to the lattice analysis introduced by von Neumann and Birkhoff.

The interpretation of a logical structure by means of a lattice is analogous to the interpretation of a formal system by means of a particular geometry. The roles of logical connectives in the physical theory can be identified with the roles of some of the operations and relations defined on the lattice associated with the theory. The result is a comprehensive view analogous to the more general understanding of geometry attained by introducing the idea of curvature. The abstract logic modeled by different lattices in different physical theories can encompass the change in the distributive law required by quantum mechanics while retaining the distributive law for the theories of classical physics.

Before proceeding with a description of lattice structures it is useful to examine more closely the assertions of the quantum theory that have precipitated such abstract logical investigations. Although quantum mechanics makes a number of puzzling assertions about physical events at microscopic scale, such as the duality of waves and particles, I shall confine my exposition to the effects of spin. The concepts required for an understanding of the quantum nature of spin reflect in an uncomplicated but nontrivial way the basic conceptual structure needed throughout quantum mechanics.



SIMPLE PHYSICAL SYSTEM made up of a box, a penny and a quarter has four states: both coins heads, both coins tails, penny tails and quarter heads, and penny heads and quarter tails. Each state can be represented by a quadrant of a disk, labeled by a four-digit binary number. Areas of the disk made up of various combinations of the quadrants correspond to every possible collection of the system's states and so to every "theoretical expression" that makes reference to some state or some combination of states. The disk is the phase space of the system.

The quantum-mechanical spin of a particle is analogous to the rotation of an ordinary object such as a top. The spin has a component along each of the three axes of three-dimensional space; the components are called the *x*, *y* and *z* components. By convention, if the fingers of the right hand curl in the same sense as the spin, the right thumb points in the spin-up direction.

The spin of a particle that carries an electric charge can be detected and manipulated by means of the magnetic moment generated by the spinning charge. The method of detection consists in passing the particle through a region in which a magnetic field varies greatly from point to point. According to classical physics, such a field gradient will deflect a moving particle by an amount that depends on the magnetic moment of the particle.

In an experiment done by Otto Stern and Walther Gerlach in 1921 silver was vaporized in a furnace and the silver atoms were directed by a series of baffles into a strongly varying magnetic field. Because the magnetic moment of the silver atoms is effectively randomized by the experimental procedure, Stern and Gerlach expected the atoms to be spread out uniformly by the magnetic-field gradient and to be detected as a smear on a photosensitive plate. They found instead that the plate recorded two well-defined patches where most of the atoms were concentrated. The magnetic moment of the atoms appeared to take on only two distinct values. The result has since been verified in many other experiments with more sensitive apparatus; in all cases the magnetic moment of an elementary particle or of a composite structure such as an atom is found to be quantized.

Because of the quantization of magnetic moment it is possible to segregate particles whose spins have a certain orientation by selecting just one of the beams that emerge from the magnetic field. The selected beam is spin-polarized in a direction parallel to the field gradient. For convenience I shall assume that the beam is moving along the *z* axis and that after passing through the magnetic field all the particles have their

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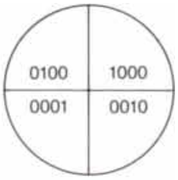
spin component along the x axis pointing up. The spin-polarized beam can then be passed through a second magnetic field at right angles to the first one and parallel to the y axis. One would suppose the beam would thereafter be polarized in both the x and the y directions. If either of the beams emerging from the second magnetic field were passed through a third field oriented in the x direction, one would expect to detect only the up value for the x component of the spin. Instead two patches again appear on the screen. The beam has somehow lost its spin-polarization along the x axis, and the measurement yields a random mixture of up and down values [*see illustration on page 204*].

This somewhat idealized experiment illustrates several fundamental principles of quantum mechanics. The passage of a beam polarized along the x axis through a magnetic field oriented so that it polarizes the y component destroys the polarization of the x component. It can be stated more generally that any operation on an elementary particle that determines the value of some quantum-mechanical variable must simultaneously randomize the value of at least one other variable; two variables that are linked in this way are said to be incompatible. If the y component of the spin is known with certainty, the value of the x component must remain completely unknown: it is therefore a maximally randomized quantity, with an equal probability (namely 1/2) of being either up or down.

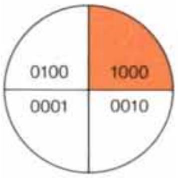
In general as the probability of measuring a particular value of one variable increases, the probability of measuring a particular value of an incompatible variable decreases. For example, if the y component of the spin became only partially polarized by the passage of the beam through the second magnetic field, the probability that the x component is up would be less than 1 but greater than 1/2. The uncertainty principle formulated by Werner Heisenberg specifies in a quantitative way how the probability of detecting a particular value of a variable depends on the probability assigned to an incompatible variable.

I shall now describe how the theoretical structure of a simple physical system can be represented by a lattice and how a logical structure can be grafted onto the lattice. Imagine a box containing a penny and a quarter and covered by a transparent lid. One can regard the box as a physical system whose state is determined by the upper face of each coin. The system can be in any one of four states: both coins heads, both coins tails, the penny heads and the quarter tails and the penny tails and the quarter heads. (The idea of discussing a system with just four possible states, although not this example, I owe to Ariadna



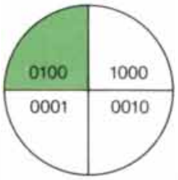
0000

P	Q	P AND NOT-P
1	1	0
1	0	0
0	1	0
0	0	0



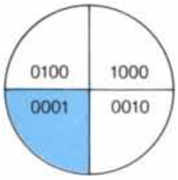
1000

P	Q	P AND Q
1	1	1
1	0	0
0	1	0
0	0	0



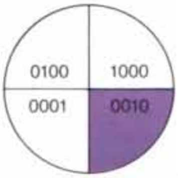
0100

P	Q	P AND NOT-Q
1	1	0
1	0	1
0	1	0
0	0	0



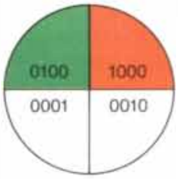
0001

P	Q	NOT-P AND NOT-Q
1	1	0
1	0	0
0	1	0
0	0	1



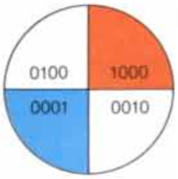
0010

P	Q	NOT-P AND Q
1	1	0
1	0	0
0	1	1
0	0	0



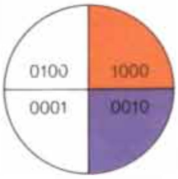
1100

P	Q	P
1	1	1
1	0	1
0	1	0
0	0	0



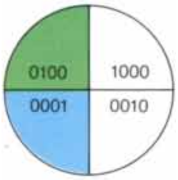
1001

P	Q	(P AND Q) OR (NOT-P AND NOT-Q)
1	1	1
1	0	0
0	1	0
0	0	1



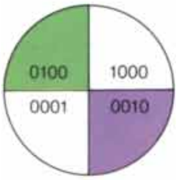
1010

P	Q	Q
1	1	1
1	0	0
0	1	1
0	0	0



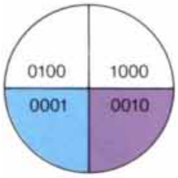
0101

P	Q	NOT-Q
1	1	0
1	0	1
0	1	0
0	0	1



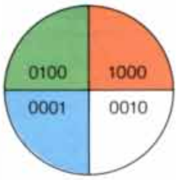
0110

P	Q	(P AND NOT-Q) OR (NOT-P AND Q)
1	1	0
1	0	1
0	1	1
0	0	0



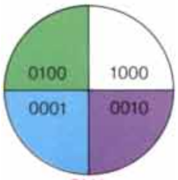
0011

P	Q	NOT-P
1	1	0
1	0	0
0	1	1
0	0	1



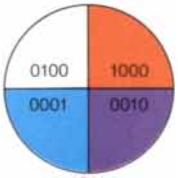
1101

P	Q	P OR NOT-Q
1	1	1
1	0	1
0	1	0
0	0	1



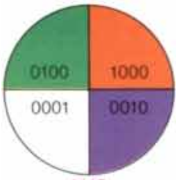
0111

P	Q	NOT-P OR NOT-Q
1	1	0
1	0	1
0	1	1
0	0	1



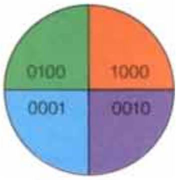
1011

P	Q	NOT-P OR Q
1	1	1
1	0	0
0	1	1
0	0	1



1110

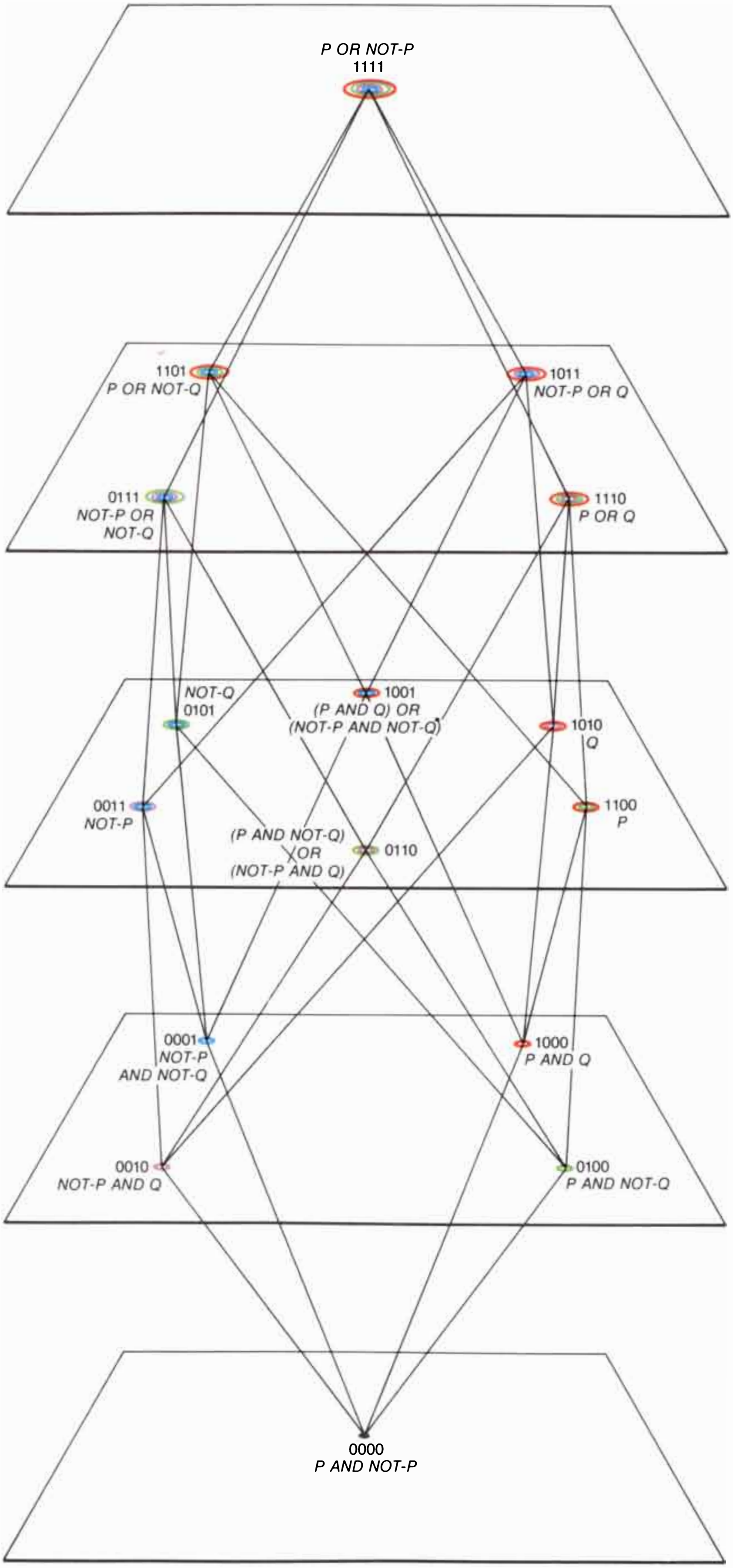
P	Q	P OR Q
1	1	1
1	0	1
0	1	1
0	0	0



1111

P	Q	P OR NOT-P
1	1	1
1	0	1
0	1	1
0	0	1

TRUTH TABLES show how the truth values of sentences can be associated one to one with the binary numbers that correspond to areas of a disk. Each quadrant of the disk represents a single state of the penny-quarter system, and every possible area is represented uniquely by the binary sum of the numbers assigned to the quadrants that constitute the disk. For any sentences *P* and *Q* a truth table shows how the truth (represented by a 1) or the falsity (represented by a 0) of a derived sentence composed of *P*, *Q* and certain logical connectives depends on the truth or the falsity of *P* and *Q* alone. There are four ways in which the sentences *P* and *Q*, taken together, can be either true or false. For each of the four ways the derived sentence can be either true or false. Hence there are 2^4, or 16, possible truth tables. For each truth table it is possible to find a sentence whose dependence on the truth values of *P* and *Q* is given by the table. For example, the sentence *P or Q* is true when either *P* or *Q* is true or both *P* and *Q* are true; it is false only if both sentences *P* and *Q* are false.



Chernavska of the University of British Columbia.)

In a simple representation each possible state of the system can be assigned a quadrant of a disk. If the disk is divided by a horizontal and a vertical line, the upper right quadrant can correspond to the state in which both coins are heads, the upper left to the state in which the penny is heads and the quarter is tails, the lower left to the state in which both coins are tails and the lower right to the state in which the penny is tails and the quarter is heads. The quadrants can be labeled in this sequence with the binary numbers 1000, 0100, 0001 and 0010. Any area of the disk that includes an integral number of quadrants can also be designated by a four-place binary number. In each place the digit is a 1 if any quadrant included in the area has a 1 in that place; otherwise the digit is a 0. For example, the area of the disk that includes all but the lower right quadrant is labeled 1101. Each quadrant must be defined so that it has no points in common with the other quadrants except the point at the center of the disk. I shall therefore assume that only the radius on the clockwise side of each quadrant belongs to that quadrant.

Any physical theory can be viewed as a statement about the possible or the actual states of a physical system. The theory may single out states or collections of states for commentary. In the penny-quarter system I shall suppose any combination of the four states can be referred to by a "physical theory" pertaining to the system. There are 16 combinations of the states, formed by combining the quadrants of the disk in all possible ways.

Suppose each combination of states is represented by a point. The 16 points can then be connected by a network of

LATTICE OF POINTS AND LINES displays the subset relations among the areas of the four-part disk. Each point represents a distinct area of the disk. The colors of the concentric circles around each point match the colors of the disk quadrants that make up the corresponding area (*see illustration on preceding page*). Points in the lattice that lie in a given plane all represent areas of the disk that include the same number of disk quadrants. The area represented by a lower point is a subset of any area represented by a higher point to which the lower point is connected by upward-moving lines. Binary numbers associated with the points correspond to the areas of the disk and to the truth tables of the sentences that label the points (*see illustration on preceding page*). The lines in the lattice also have a logical interpretation. Wherever they represent subset relations for the areas of the disk, they also represent entailment relations. A sentence represented by a lower point entails all sentences represented by higher points that can be reached by upward-moving lines.

lines to show how some areas of the disk can be considered subsets of other areas [*see illustration on opposite page*]. A line connecting two points indicates that the lower point is a subset of the higher point. More precisely, the area represented by the lower point is included in the area represented by the higher point.

The lowest point in the network represents the zero area (0000), the area of the point at the center of the disk that all the quadrants have in common. From the zero point, lines in the network travel upward to the four points (1000, 0100, 0001 and 0010) that represent the areas of each of the four quadrants of the disk. As the connecting lines indicate, each of these areas includes the central point as a subset. Above the four points representing the quadrants are points representing larger areas of the disk, each of which includes at least two quadrants and the central point. At the top of the network is the point 1111, which represents the area of the entire disk. Subset relations can be indicated not only by a direct upward line but also by a line that passes through intermediate points. Thus the uppermost point is linked to all the points below it. One can also think of each point as being connected to itself.

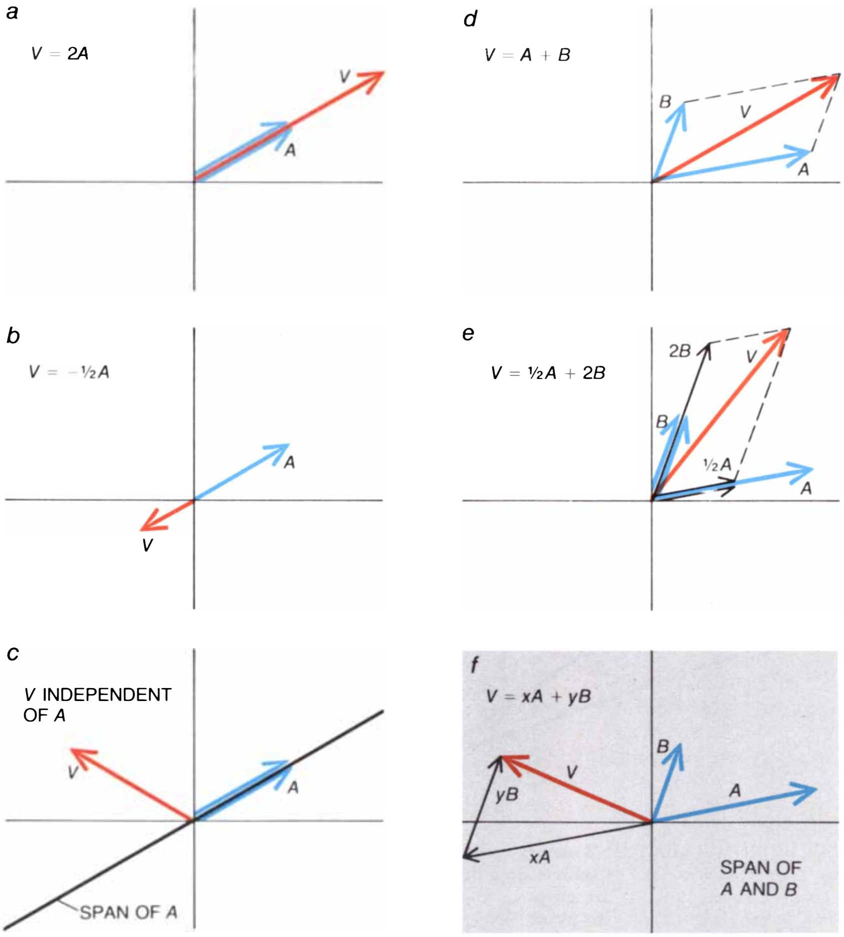
It is now possible to define two operations on the network of points and lines that make the network a lattice of the kind defined by von Neumann and Birkhoff. Given any two points *a* and *b* in the network, the lowest point in the lattice to which *a* and *b* are both connected by lines moving upward from at least one of them is referred to as *a-join-b*. If both *a* and *b* are directly linked to the same higher point, that point is *a-join-b*. If *a* and *b* are connected to each other, then *a-join-b* is the higher of the two points. For the lattice representing areas of the disk the *join* of two areas is the point representing the smallest area that includes both of them; in set theory the same concept is called the union. The *join* of 0000 (the central point of the disk) and 1000 (the upper right quadrant) is 1000, since the quadrant includes the central point. The *join* of 1000 and 0100 (the upper left quadrant) is 1100: the upper half of the disk.

A second operation on the lattice is defined by selecting the highest point to which both *a* and *b* are connected by lines moving downward from at least one of them; such a point is called *a-meet-b*. On the disk the *meet* of two areas is the largest area they have in common; in set theory it is called the intersection. The *meet* of 1100 (the upper half) and 1010 (the right half) is 1000: the upper right quadrant. The *meet* of 1000 and 0000 is the point 0000.

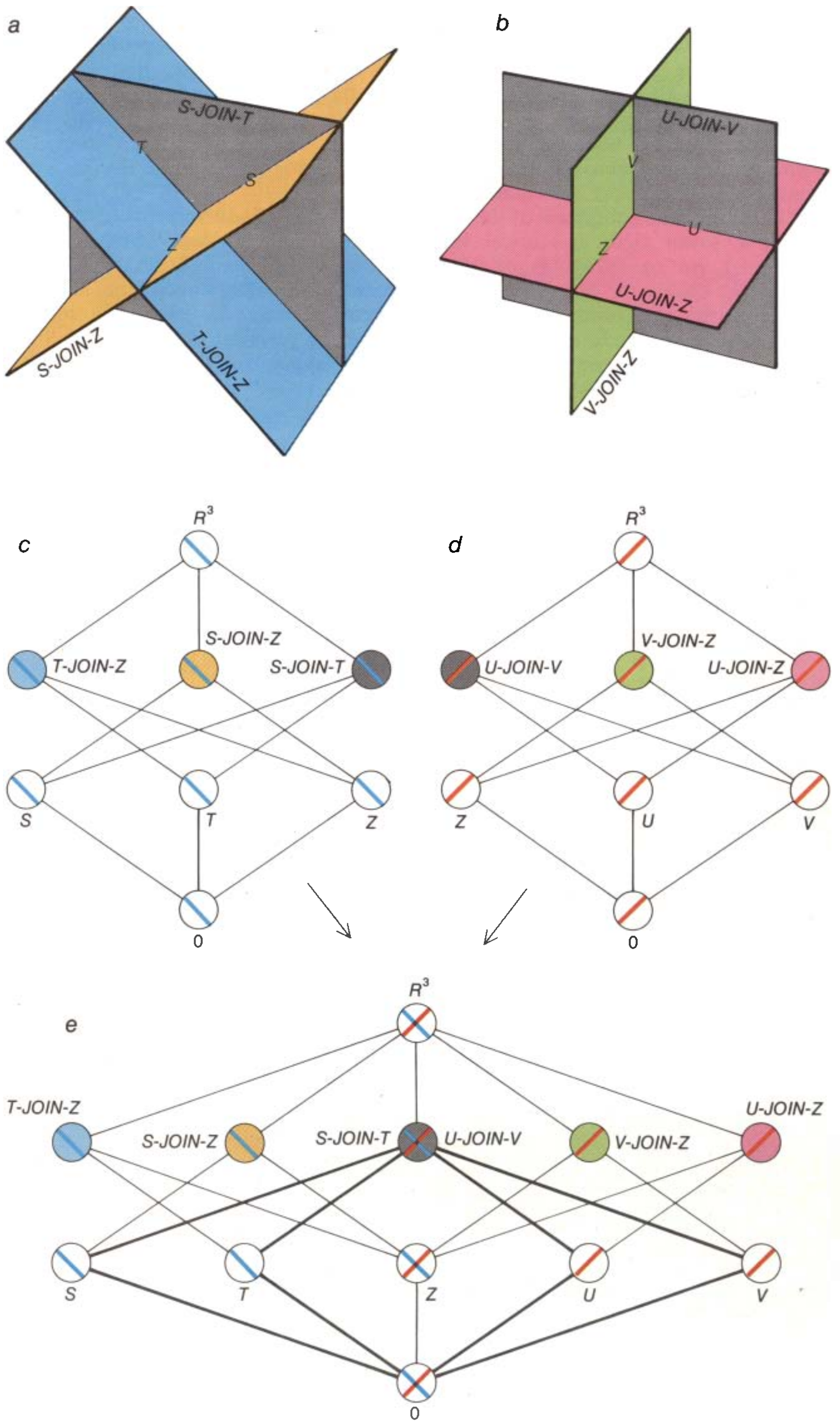
The lattices I shall discuss are called complemented ones: for any point *a* in the network it must be possible to find its complement *a'*, which has the property that *a-join-a'* is the top point of the network (1111) and *a-meet-a'* is the bottom point (0000). On the disk the complement of an area is another area that has no points except the center of the disk in common with the first area, but whose area taken together with the first area makes up the entire area of the disk. The points 0100 (the upper left quadrant) and 1011 (the area composed of the upper right, lower right and lower left quadrants) are complements of each other. The analogous operation in set theory is also called complementation.

The behavior of the lattice operations *join* and *meet* depends on the lattice on which they are defined. For the lattice representing the areas of the disk, however, they function in a familiar way. They obey the same laws as their counterparts in set theory: union and intersection. In particular the distributive laws hold. For any points *a, b* and *c* in the lattice, *a-meet-*(*b-join-c*) is the same point as (*a-meet-b*)*-join-*(*a-meet-c*). Similarly, *a-join-*(*b-meet-c*) is the same point as (*a-join-b*)*-meet-*(*a-join-c*). The resulting mathematical structure is called a complemented distributive lattice; it is also known as a Boolean algebra, after the 19th-century British logician George Boole.

I have described the properties of an abstract representation of the penny-quarter system, namely the set relations among the areas of a disk. What kinds of statement might be employed to describe the physical system itself? Suppose *P* stands for the statement "The penny is heads," *Q* stands for "The quarter is heads," *not-P* stands for "The penny is tails" and *not-Q* stands for "The quarter is tails." Any property of the penny-quarter system that a physicist might want to describe by means of a theory about the system can then be defined by writing a compound sentence. For example, the compound sentence *P*



SPAN OF A VECTOR SPACE is the set of all vectors that can be obtained by adding any vectors in the space or by multiplying them by a scalar quantity (a quantity that has only a magnitude and not a direction). The span of the single vector *A* (*blue arrows*) is a straight line passing through *A*, because additional vectors (*red arrows*) can be generated only by scalar multiplication or by the addition of vectors pointing along the line (*panels* a *and* b). Vectors that cannot be generated by such operations are independent vectors (*panel* c). Two independent vectors *A* and *B* combine by vector addition and scalar multiplication according to the parallelogram law (*panels* d *and* e). Their span is a plane because any point in the plane can be reached by vectorially adding some scalar multiple of *A* to some scalar multiple of *B* (*panel* f).



INCLUSION RELATIONS among vector subspaces of the three-dimensional physical space (R^3) can be represented by lattices. In *a* the subspaces represented by the lines *s* and *t* span the subspace represented by the plane *s-join-t*. Similarly, the lines *s* and *z* span the plane *s-join-z* and the lines *t* and *z* span the plane *t-join-z*. In *b* the lines *u*, *v* and *z* span corresponding planes, where the lines *u* and *v* are in the same plane as the lines *s* and *t*. The points in lattices *c* and *d* represent the vector subspaces formed by the lines and planes, by the zero subspace (the single point where the three lines in each diagram intersect) and by the vector space R^3 itself. Vectors in a subspace represented by a lower point in a lattice also lie in any subspace represented by a higher point to which the lower point is connected by upward-moving lattice lines. The two lattices can be linked at points that represent the same subspaces, namely the points R^3, *s-join-t* (equivalent to the point *u-join-v*), *z* and 0. The resulting lattice (*e*) is a nondistributive one, identical with the lattices at the right in the illustration on page 203. The compound lattice displays the inclusion relations that would hold among the vector subspaces if the two planes *s-join-t* and *u-join-v* were coincident and the lines *s, t, u, v* and *z* all crossed at the origin.

and Q makes the assertion "Both coins show heads." The sentence *not-P or Q* says that in the present state of the system "Either the penny is showing tails or the quarter is showing heads"; in other words, it specifies that the state is anything but the penny showing heads and the quarter showing tails.

In ordinary circumstances (unlike those encountered in quantum mechanics) logical intuition is a fairly reliable guide to the ways in which the truth or falsity of a compound sentence depends on the truth or falsity of its constituents. The sentence *P and Q* is true only if both *P* is true and *Q* is true, and hence only if both coins show heads; otherwise the compound sentence is false. The truth value of the compound sentence must be evaluated separately for each possible combination of truth and falsity of the constituent sentences.

A complete account of how the truth value of a compound sentence is determined by its constituent sentences is called a truth function, and logicians customarily display the evaluation of the truth function in an array called a truth table. Suppose 1 stands for true and 0 for false. Then any compound sentence made up of the two constituent sentences *P* and *Q* can be represented as a four-digit binary number in the same way as the areas of the disk were classified. There is a 1 or a 0 in the truth table for each of the four possible pairs of truth values for *P* and *Q*. Thus the truth function for every possible sentence having two constituents is given by one of 16 possible truth tables. The truth values of the sentence *P and Q*, for example, are given by the truth function 1000; the sentence is true only in the quadrant of the disk labeled 1000.

The relation between the truth function of a compound sentence and the number for the area of the disk in which the compound sentence is true is a general one. The binary numbers are the same [*see illustration on page 207*]. Hence to each area of the disk, and so to each point on the lattice, one can attach some sentence made up of the elements *P, Q, not, and* and *or*. More strictly one can associate with each point in the lattice an equivalence class of sentences, made up of all the sentences that have the same truth table. The sentence *not-(P and Q)*, for example, is associated with the same point in the lattice as the sentence *not-P or not-Q*.

The most important outcome of the association of sentences with lattice points is that it shows how the lattice can display logical relations. If the sentences *A* and *B* are represented by the points *a* and *b* in the lattice, then the sentences *A and B* and *A or B* are represented respectively by the points *a-meet-b* and *a-join-b*. Moreover, the lines connecting lower points in the lattice to higher points represent the logical relation of entailment.

Sentence A entails sentence B provided that B is true whenever A is true. The truth table for B must include a 1 on every line where a 1 appears in the truth table for A. (The truth table for B may include other 1's as well.) Therefore entailment on the lattice that describes the logical structure of the penny-quarter system is represented by the same lines that represent the subset relation.

The lattice for the penny-quarter system shows the same logical relations as the lattice that corresponds to Newtonian, or classical, mechanics. The state of a Newtonian particle is characterized by the position of the particle at a specified time and by its momentum (the product of its mass and its velocity). For simplicity suppose the Newtonian system consists of a single particle constrained to move in only one dimension. Just as a state of the penny-quarter system can be represented by an area on a disk, so the state of the Newtonian particle can be represented by a point on a plane. The coordinate axes of the plane are labeled with values of position and momentum, so that every point on the plane corresponds to some pair of values for position and momentum. The plane is called the phase space of the system.

What is the mathematical structure of the phase space to which the simplified Newtonian theory conforms? The statements of Newtonian physics make reference to regions, or subsets, of the phase space, just as statements about the penny-quarter system make reference to areas of the disk. A statement corresponds to a region of the plane whenever the statement holds true for every point in the region and for no other points. For example, the statement that a particle has position a corresponds to a line of points in the phase space; the line is a units from the origin along the position axis and parallel to the momentum axis.

The class of regions of Newtonian phase space is an infinite class, and so the corresponding lattice has an infinite number of points. In other respects, however, the mathematical structure of the lattice is the same as the structure of the lattice for the penny-quarter system. The regions of the phase space and hence the points in the Newtonian lattice are ordered by set inclusion, and as before the operations on the lattice are equivalent to set union and set intersection. Moreover, the logical relations defined by the structure of the lattice and by the operations *meet* and *join* remain the relations of classical logic. The distributive laws are valid. The extension of the phase space and the lattice to three-dimensional Newtonian systems with many particles is complicated but not different in principle from the one-dimensional, single-particle case.

In quantum mechanics the situation is altogether different. The state of a particle is no longer specified by its position and its momentum because position and momentum are incompatible variables, which cannot be determined simultaneously. Instead the state of a particle is defined by the theoretical construct called a wave function, which gives the probability of finding that the particle has a certain value of a physical variable. For example, the spin state of an electron is given by a wave function that specifies the probabilities that the x, y and z components of the spin are either up or down.

Consider the y component of the spin. The probability that the y component is spin-up can vary from 1 to 0 while the probability that it is spin-down varies from 0 to 1. For any spin state the sum of the two probabilities is equal to 1. It is mathematically convenient to represent the spin state as a vector: a quantity that has both a magnitude and a direction. The spin vector is the vector sum, calculated according to the parallelogram law of vector addition, of two other vectors. The latter two vectors lie along two perpendicular lines that represent the two possible values the spin can assume along the y axis. They are called the spin-up and the spin-down probability amplitudes. The square of the magnitude of the vector representing the spin-up probability amplitude gives the probability of finding the y component of the spin in the up state. Similarly, the square of the magnitude of the vector corresponding to the spin-down probability amplitude gives the probability of finding that the y component of the spin-state vector points down [*see top illustration on page 205*].

The abstract vector space with its perpendicular axes labeled "spin-up probability amplitude" and "spin-down probability amplitude" is a phase space of quantum mechanics. By superposing on the phase space an additional pair of perpendicular axes at a 45-degree angle to the first pair, it is possible to represent the essential features of the uncertainty principle as it applies to spin [*see bottom illustration on page 205*]. The additional axes can be designated the spin-up and spin-down probability amplitudes for the x component of the spin, just as the original axes designate the amplitude of the y component. If the spin-state vector lies along the spin-up axis for the x component (and so predicts the result of a measurement of the x component with certainty), the vector lies halfway between the spin-up and the spin-down axes for the y component. The length of the projection of the vector onto the spin-up axis for the y component is therefore $\sqrt{2}/2$, and so the probability of finding that the y component of the spin is up is the square of $\sqrt{2}/2$, or 1/2. Similarly, the probability of finding that the y component of the spin is down is also 1/2. Thus, in accord with the principle of uncertainty, there is no spin state in which a probability of 1 can be assigned both to some value of the x component and to some value of the y component.

The kind of inclusion relation that is defined on a vector space is the relation of being a subspace, rather than of being merely a subset, of the given vector space. It is the subspaces of a vector space that correspond to the propositions of the quantum theory.

What is a vector subspace? The subspace must itself be a vector space. The most important property of a vector space is that adding any two vectors in the space generates a third vector that lies in the same space. Similarly, multiplying any vector by a scalar quantity (one that has only a magnitude, not a direction) also yields another vector in the space. In mathematical terms the vector space is said to be closed under vector addition and the multiplication of a scalar.

The closure criterion can be employed to construct a vector space. The complete vector space can be generated from vectors called basis vectors if every vector in the space can be obtained by adding the basis vectors according to the parallelogram rule, perhaps after each basis vector has been multiplied by a suitable constant. (Scalar multiplication can lengthen or shorten a vector and can make it point in the diametrically opposite direction, but it cannot otherwise alter its direction.) The vector space generated in this way by two basis vectors A and B is called the span of A and B [*see illustration on page 209*].

A subspace is said to be a proper subspace if there is at least one vector in the vector space that does not lie within the span of any basis vectors of the subspace. For example, the u-v plane is a vector space that can be spanned by two vectors directed along the positive u and v axes. There is no way to add the two vectors to get a third vector that points out of the u-v plane. Hence the u-v plane is a proper subspace of the vector space constituted by three-dimensional space.

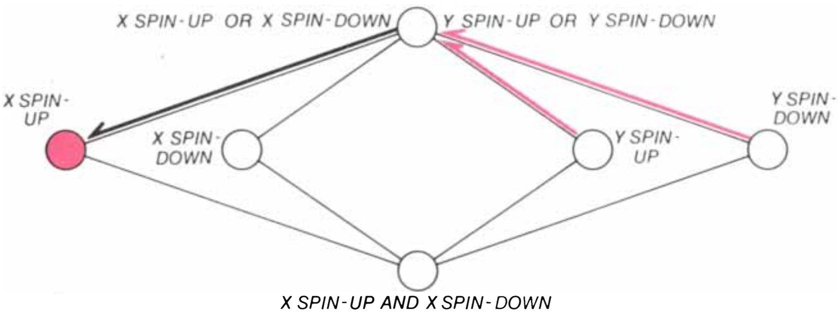
In order to form a lattice whose points correspond to the propositions of quantum mechanics one must form the lattice of vector spaces and their proper subspaces. Consider three-dimensional physical space with a designated point 0 as the origin. The subspaces of the space include the origin itself, all straight lines that can be drawn through 0, all planes that include 0 and the entire three-dimensional space. Some of the subspaces are ordered by inclusion, so that logical entailment on the lattice can still be a matter of subspace inclusion.

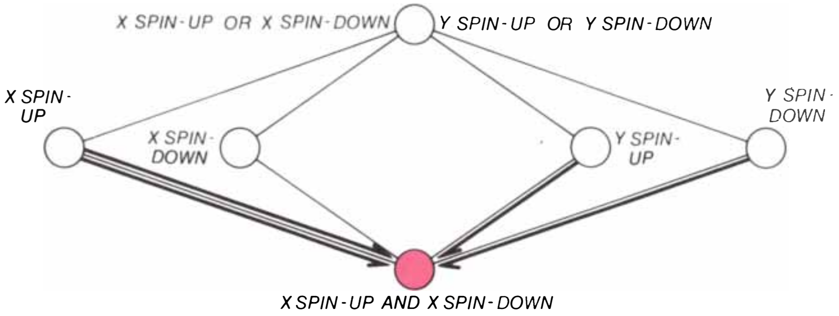
A crucial difference, however, is that the *join* operation on the lattice of subspaces is not the union operation of set theory but rather the span of two subspaces. For example, the *u* axis and the *v* axis are both one-dimensional subspaces. Their union is the set of all the vectors that begin at the origin and terminate at any point on one of the two lines. The set is not a subspace, however, because it is not closed under vector addition. Therefore the union of the axes cannot be represented by a point in the lattice of subspaces. The *u-v* plane is a subspace, namely the span of the two one-dimensional subspaces. Moreover, the span of *u* and *v* is the smallest subspace that includes all the vectors on both lines. The point in the lattice of subspaces that is the result of the operation *u-join-v* is therefore the point that represents the *u-v* plane.

Because of the peculiar properties of the operation of vector spanning, the lattice of subspaces is not distributive. Suppose the lattice includes points representing the four lines *s, t, u* and *v;* all the lines lie in the *u-v* plane and they all pass through the origin. Then *s-join-t* and *u-join-v* are both represented by the same point in the lattice, namely the point that corresponds to the *u-v* plane.

The point *s-meet-*(*u-join-v*) is by definition the largest subspace that *s* and *u-join-v* have in common; since *u-join-v* is the *u-v* plane and *s* lies in the plane, the largest subspace they have in common is the line *s* itself. On the other hand, *s-meet-u* and *s-meet-v* both include only the origin, or point 0 on the lattice, and the *join* or span of these expressions—that is, (*s-meet-u*)*-join-*(*s-meet-v*)—is also the point 0. It follows that the point designated *s-meet-*(*u-join-v*) and the point (*s-meet-u*)*-join-*(*s-meet-v*) are not identical in the lattice of vector subspaces.

If the logical connectives *and* and *or* and the relation of entailment are defined on the lattice of subspaces as they were on the lattice of subsets for the penny-quarter system, the structure of the lattice is the same as the structure of nondistributive quantum logic. The subspace *s* can be identified with the sentence "The *x* component of the spin is up" and the subspace *t* with the sentence "The *x* component of the spin is down." Similarly, the subspaces *u* and *v* correspond to the analogous sentences for the *y* component. The compound sentence "The *x* component of the spin is up and the *y* component is either up or down" is associated with the point in the lattice *s-meet-*(*u-join-v*). The compound sentence "Either the *x* component of the spin is up and the *y* component is up or the *x* component of the spin is up and the *y* component is down" is associated with the point (*s-meet-u*)*-join-*(*s-meet-v*). Since these lattice points are not identical, the logical structure associated with the lattice is nondistributive. Moreover, the lattice shows that (*s-meet-u*)*-join-*(*s-meet-v*) entails *s-meet-*(*u-join-v*) but not vice versa [*see illustration on this page*].





LOGICAL RELATIONS among the statements describing the spin of a particle can be modeled in a highly simplified form by lattices identical with the one outlined in heavy black lines at the bottom of the illustration on page 210. The lattice points correspond to sentences describing pure spin states. On each lattice the word *and* is identified with the *meet* of two points (*black arrows*), and the word *or* is identified with the *join* of two points (*colored arrows*). The sentences below the lattices, which are logically equivalent according to the distributive law, are modeled by the operations shown by the arrows. Because the structure of the lattices derives from the structure of the inclusion relation for vector spaces, the lattices are nondistributive. The two compound sentences are not represented by the same point in the lattice. Hence the lattices model the failure of the distributive law of logic to hold for statements about spin.

Confronted by the success of lattice theory in modeling logical relations within various physical theories, one tends to forget that the approach presupposes the resolution of an important philosophical issue. A dominant theme in philosophy during the past 200 years has been the thesis that there are two kinds of true statement: contingent facts about the world and truths of logic that would hold no matter what the condition of the world. The Scottish philosopher David Hume called the two kinds of statement "matters of fact" and "relations of ideas." The notion that the quantum theory might call for a revision of logic presupposes a denial of Hume's thesis that there are two kinds of truth.

Nevertheless, philosophers are now inclined to agree with Willard Van Orman Quine of Harvard University that no sharp distinction of this kind can be maintained. The laws of logic, Quine argues, have a central place in our web of beliefs, but a strong pull on that web at the periphery of observation can cause even the center of the web to become distorted. Logical considerations cannot themselves justify a revision of logic (how could they?), but one can (again to adopt a metaphor of Quine's) gerrymander the language when the fit becomes too strained between the natural world and the language we have inherited to describe it.

Even if it is granted that logic can be revised, there are many possible responses to quantum logic. One extreme is to deny that quantum logic is a logic and to say that what is going on is merely algebra under another name. The other extreme is to say that because quantum mechanics deals with particles that are the fundamental constituents of the universe, one should replace classical logic with quantum logic and learn to "think quantum logically," hard though that might be.

With respect to the first response one can say the following. Logic, although notoriously hard to define, deals with certain kinds of relations between sentences: what follows from what, what is consistent with what, and so on. Quantum logic does this too. What makes quantum logic peculiar is that it deals entirely with sentences stating that some vector lies in some subspace. The oddi-

CLASSICAL TWO-VALUED LOGIC

P	T	T	F	F					
Q	T	F	T	F					
NOT-P	F	F	T	T					
NOT-Q	F	T	F	T					
P OR Q	T	T	T	F					
P AND Q	T	F	F	F					
P IMPLIES Q	T	F	T	T					
P IS EQUIVALENT TO Q	T	F	F	T					

REICHENBACH'S THREE-VALUED LOGIC

P	T	T	T	I	I	I	F	F	F
Q	T	I	F	T	I	F	T	I	F
NOT-P (CYCLICAL)	I	I	I	F	F	F	T	T	T
NOT-P (DIAMETRICAL)	F	F	F	I	I	I	T	T	T
NOT-P (COMPLETE)	I	I	I	T	T	T	T	T	T
P OR Q	T	T	T	T	I	I	T	I	F
P AND Q	T	I	F	I	I	F	F	F	F
P IMPLIES Q (STANDARD)	T	F	F	T	T	T	T	T	T
P IMPLIES Q (ALTERNATIVE)	T	F	F	T	T	T	T	T	T
P IMPLIES Q (QUASI)	T	I	F	I	I	I	I	I	I
P IS EQUIVALENT TO Q (STANDARD)	T	I	F	I	T	I	F	I	T
P IS EQUIVALENT TO Q (ALTERNATIVE)	T	F	F	F	T	F	F	F	T

THREE-VALUED LOGIC proposed by Hans Reichenbach in 1944 maintains that some sentences describing quantum-mechanical phenomena are neither true nor false but indeterminate. For example, if the *x* component of the spin of a particle is known to be up, a statement asserting that the *y* component has a specific value would be classified as indeterminate in Reichenbach's scheme. Instead of focusing on the distributive law, Reichenbach defined enriched patterns of logical connectives by means of truth tables. The tables assign one of the three truth values—true (*T*), false (*F*) or indeterminate (*I*)—to each of the nine possible combinations of truth values for the sentences *P* and *Q*. Ten truth functions are designated as having a special place in the logic of quantum mechanics. The truth functions of ordinary two-valued logic are special cases of the extended truth functions (*color*). In three-valued logic there are 3^9, or 19,683, possible truth functions for derived sentences with two constituents.

ties of quantum logic are consequences of two requirements that must be met by any sentence to which the logic pertains. First, the sentences must be among those that ascribe quantum-mechanical properties to individual systems. Second, when two such sentences are linked by means of the quantum-mechanical analogues of *and* and *or*, the resulting sentence must still be descriptive of the physical system. In dealing with this closed set of sentences the logical relations among them are not those of classical logic.

Is one therefore forced to adopt the second stance and think quantum logically? In part the answer is yes. Certain kinds of statements met with in a physical theory fit together in ways that do not conform to classical logic. By no means, however, does this finding imply that classical logic should everywhere be replaced by quantum logic. Such a drastic revision of everyday ways of thinking could be justified only if it brought with it a vast simplification of the overall theory of the world; it is doubtful that such a simplification would be achieved. Even within quantum mechanics the logico-algebraic approach, although valuable, has not cleared away all the perplexities. Furthermore, even if quantum logic were totally successful in its own domain, the extension of it to other realms would seem peculiar. One arrives at quantum logic by considering the mathematical structure of the formalism of quantum mechanics. That mathematical structure, however, is based on the deductive patterns of classical logic. Hence classical logic is presupposed in the development of quantum logic.

One is left, therefore, with a family of logics that includes classical logic, quantum logic and perhaps other logics as well. Among them one logic still has priority. Although the nonclassical logics may have specialized applications, the logic employed for abstract reasoning, including reasoning about logic, will probably continue to be classical logic.